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Reply to "Comment on 'Time-averaged properties of unstable periodic orbits and chaotic orbits in ordinary differential equation systems' "

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We point out that the original work [Saiki and Yamada, Phys. Rev. E **79**, 015201(R) (2009)] was aimed at addressing the question of why even a single periodic orbit of low period can capture averaged properties of physical quantities reasonably well in some dynamical systems including turbulent fluids. This point of view is quite different from that of the Comment [Zaks and Goldobin, Phys. Rev. E **81**, 018201 (2010)].

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It is known that physical quantities of a chaotic state can be well estimated from a weighted average of a set of unstable periodic orbits (UPOs) embedded in a chaotic attractor by using a cycle expansion theory [1]. For the case of the Lorenz system with a set of classical parameter values, Franceschini *et al.* reported that Hausdorff dimension can be estimated from a set of UPOs [2]. Zoldi [3] reported that x-variable histogram along a chaotic orbit can be estimated from a set of UPOs. The Comment [4] is in the line of these works, which approximate a chaotic statistics in high precision from the information of many UPOs. From these works, it is natural that the weighted average of time-averaged values along UPOs approximate the mean property of chaotic segments with the corresponding time length.

Meanwhile, it is recently reported in the literature of fluid turbulence that only one UPO with a relatively low period can capture some significant statistical properties of turbulence ([5,6]). Many researchers including us are surprised at the fact that even a single UPO with a relatively low period can capture chaos properties, because it appeared that we have to employ many UPOs including long UPOs to capture chaos properties even in a rough sense. Motivated from the two papers, some researchers try to find the answer why only one UPO with a low period can capture mean properties of chaotic states. This is in the different line from the above. That is, our interest was not to study chaotic distribution with a weighted average of UPOs, but to explain the reason why only a few UPOs with relatively low periods can capture chaos properties.

In our rapid communication [7], we studied time-averaged values along UPOs without weighted average in comparison with those along chaotic segments, for three chaotic systems described by ordinary differential equations (ODEs) [Lorenz system, Rössler system, and Economic system]. Our main result is that for all of three systems, time-averaged values along UPOs with low periods are more localized around time-averaged value of a chaotic state than time-averaged values along chaotic segments with the corresponding time length, and that the distributions of time-averaged values along UPOs with lower periods and higher periods have similar shapes. This implies that one of any UPOs with low periods can capture chaos properties at least in a rough sense as in the turbulence studies, and that a longer UPO does not necessarily give a better approximation of the chaotic mean properties. It should be remarked that in a mathematical language our interest lies not in the physical invariant measure but in the maximal entropy measure with the equivalent weight for each UPO.

The Comment mentions the limiting property of distribution of time-averaged values along UPOs with cycle-N, which we stated as a conjecture. Even in our calculation for the Lorenz system standard deviation (STD) of the distribution of time-averaged value of z along UPO with cycle-N seems to decrease when we focus our attention to $10 \leq N \leq 14$, although it seems to increase for $N < 10$. In this sense, the result mentioned in Ref. [7] as "In fact, even for the set of unweighted UPOs, the standard deviation of $\langle z \rangle$ decays as $\sim 0.74/\sqrt{N}$" is an important supplemental result in the line of our rapid communication. But it would be difficult to see whether the distribution converges to a deltalike distribution or not with certainty from numerical results, and we just state the limiting property of the distributions as a conjecture.

Comment also mentions the topological entropy. Mathematically topological entropy $h_{\text{top}}(f)$ is defined by

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \frac{1}{n} \log s(n, \epsilon), \tag{1}$$

where $s(n,\epsilon)$ is the maximum number of points with ϵ-distinct orbits of lengths n. We say that x and y have ϵ-distinct orbits of lengths n if $d[f^j(x), f^j(y)] > \epsilon$ for some $0 \leq j < n$. $h_{\text{top}}(f)$ equals to $\limsup_{n\to\infty}\{\log(\text{number of periodic orbits with cycle } n)/n\}$ only in some limited cases like Axiom A diffeomorphisms. In most cases, we only know that $h_{\text{top}}(f)$ is bounded by $\limsup_{n\to\infty}\{\log(\text{number of periodic orbits with cycle } n)/n\}$. In fact, it is found by Kaloshin [8] that there are maps whose periodic points grow super exponentially and that it is a generic property. For this kind of system, we cannot calculate topological entropy by the exponential growth of the number of periodic points, as it is known that topological entropy is finite. If we restrict ourselves to rigorous mathematical discussions, we cannot say almost anything about topological entropy for almost all physical systems, as we know neither whether the topological entropy can be calculated from the growth rate of the number of periodic points or not nor the

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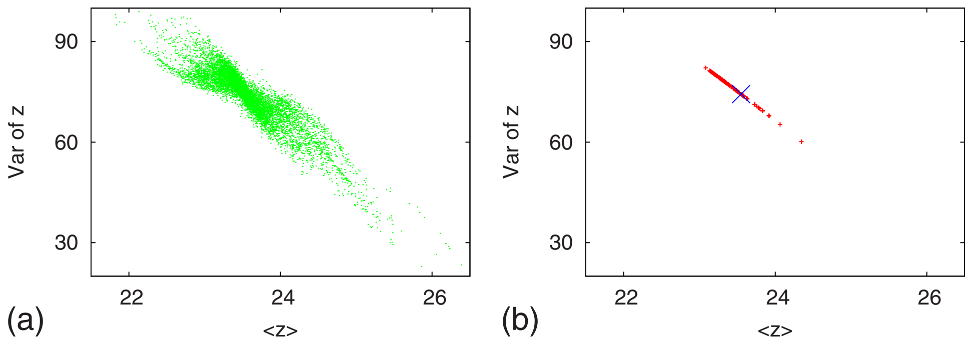

FIG. 1. (Color online) Var of z vs. $\langle z \rangle$ for the Lorenz system [(a) Chaotic segments (N=11) [dot (green online)], (b) UPO (N=11) [small cross (red online)], and Chaos [big cross (blue online)].

explicit expression of the symbolic dynamics for almost all physical systems including three systems studied in our rapid communication. However, traditionally in physical literature, as we did in our rapid communication of Physical Review E [7], the topological entropy is "estimated" from the growth rate of the number of covered periodic points with low periods, as the value is thought to be a significant index related to topological complexity. From the estimated values for three systems, it is implied that the systems have relatively low topological complexities.

The Comment authors doubt the difference between time-averaged values along UPOs and chaotic segments with the corresponding time lengths. Here, we show an evidence which supports our assertion. Figure 1 shows the relation of $\langle z \rangle$ and the Variance of z along segments of chaotic orbits (Chaotic segments) (N=11), UPOs (N=11) and a chaotic orbit (Chaos). It shows that whereas points for Chaotic segments are scattering in ($\langle z \rangle$, Variance of z) plane [Fig. 1(a)], the points for UPOs are almost on a straight line [Fig. 1(b)]. In addition, the point for a chaotic orbit is on the line [Fig. 1(b)]. (The detailed results on this issue will be reported elsewhere [9].) In summary, our rapid communication discussed a question why only a few UPOs with relatively low periods can capture some dynamical quantities along a chaotic orbit. We investigated distribution of time-averaged values along UPOs with relatively low periods. The main result was that by using more than 1000 UPOs, STD of dynamical quantities along UPOs in three ODE systems are more localized around mean quantities of chaotic states than those along chaotic segments with the corresponding time lengths. The property is conjectured to be in the background of successes in turbulence studies in which only a few UPOs with relatively low periods can capture turbulence statistics. The Comment is in a different line of important works from ours, that is, it tries to get an approximation of chaotic averages by using many UPOs with weighted averages.

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